JOURNAL OF APPROXIMATION THEORY 63, 39–55 (1990)

Eigenvalues of Integral Operators with Positive Definite Kernels Satisfying Integrated Hölder Conditions over Metric Compacta

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Communicated by P. L. Butzer

Received December 13, 1988

We determine the asymptotic eigenvalue behaviour of integral operators generated by positive definite kernels satisfying an integrated Hölder condition on metric compacta. We also show that this behaviour is the best possible. © 1990 Academic Press, Inc.

INTRODUCTION

The aim of this paper is to determine the asymptotic eigenvalue behaviour of integral operators in $L_2(X, \mu)$ whose kernels are positive definite and satisfy a certain Hölder-continuity condition. Here X is a compact metric space and μ is a finite Borel measure on X.

The study of integral operators over compact metric spaces was initiated by a problem posed by Pietsch at the sixth Polish–GDR seminar on "Geometry of Banach Spaces and Operator Ideals" (Georgenthal, 1984). Since then several papers have appeared dealing with operators of this kind, see [5, 6, 1] and also the forthcoming monograph [2]. The obtained eigenvalue results reflect compactness properties of the underlying space X expressed in terms of its entropy numbers $(\varepsilon_n(X))$. Moreover, it turned out

* Supported in part by the Spanish Ministerio de Educación y Ciencia (Programa "Estancia de Científicos y Tecnólogos Extranjeros en España").

0021-9045/90 $3.00

that there are connections between these eigenvalue results and dimension theory of compact metric spaces, see [1].

In all these papers continuity of the kernel in both variables and α-Hölder continuity in one variable is assumed. We consider here weaker integrated α-Hölder conditions and do not require global continuity.

In the one-dimensional case, i.e., $X = [0, 1]$ and μ = Lebesgue measure, this problem was first considered by Reade [9]. He showed that the decay of the eigenvalues of such a positive integral operator is of order $O(n^{-\alpha-1})$. Later on, Cochran and Lukas [3] extended this result to the case when a higher-order derivative of the kernel satisfies that Hölder condition. They also gave a new proof of Reade's result. But in both of these papers the additional assumption of global continuity of the kernel is necessary for the methds of proof.

Thus, even in this one-dimensional case, we improve the known result since we do not require global continuity of the kernel, which seems a bit artificial in connection with integrated Hölder conditions.

The general approach that we develop allows us to obtain as an immediate consequence a generalization of Reade's result to the multi-dimensional case of bounded domains in $\mathbb{R}^N$. The procedure we use also works for arbitrary Borel measures and not only the Lebesgue measure.

Nevertheless, the basic idea of our proof is not new. One can find it in several papers by different authors (in particular, in [9, 3]). We first approximate the integral operator by a finite rank operator such that the difference is still a positive operator in $L_2(X, \mu)$, and then we estimate from above the trace of this difference.

The paper is organized as follows. In the first section we fix notation and give some preliminaries. In the second section we prove the eigenvalue result, and in the third one we show that it is optimal in general. In the final fourth section we construct some examples of compact metric spaces with regular entropy behaviour. Thus we illustrate that the condition $\varepsilon_{2n}(X) \simeq \varepsilon_n(X)$, which we need for our results, is quite natural and is satisfied for many compact spaces.

1. Preliminaries

In what follows, we designate by X a compact metric space and by d its metric. The symbol $B(x, \varepsilon)$ stands for the *open ball* of radius $\varepsilon > 0$ with centre $x \in X$

$$B(x, \varepsilon) = \{y \in X: d(x, y) < \varepsilon\}.$$

Given a set $A \subseteq X$,

$$\mathrm{diam}(A) = \sup\{d(x, y): x, y \in A\},$$

denotes its *diameter*. We say that the set A is *ε-distant*, if $d(x, y) \geqq \varepsilon$ for all $x, y \in A$, $x \neq y$.

The *entropy numbers* $(\varepsilon_n(X))$ of X are given by $\varepsilon_n(X) = \inf\{\varepsilon > 0$: there are $x_1, ..., x_n \in X$ with $X = \bigcup_{i=1}^n B(x_i, \varepsilon)\}$ (see [5]). Note that $\lim_{n \to \infty} \varepsilon_n(X) = 0$ is equivalent to precompactness of X (and therefore to compactness if X is complete). Thus the rate of decay of $\varepsilon_n(X)$ as $n \to \infty$ can be considered as a measure for the "degree of compactness" of X.

By a *partition* of X we mean a finite family $\mathscr{A}$ of disjoint Borel sets $A_1, ..., A_n$ such that $\bigcup_{i=1}^n A_i = X$. The diameter of $\mathscr{A}$ is defined by

$$\mathrm{diam}(\mathscr{A}) = \sup\{\mathrm{diam}(A): A \in \mathscr{A}\}.$$

We say that the partition $\mathscr{A}$ is finer than the partition $\mathscr{B}$ ($\mathscr{A} \prec \mathscr{B}$) if each $A \in \mathscr{A}$ is contained in some $B \in \mathscr{B}$. Given any two partitions $\mathscr{A}_1$ and $\mathscr{A}_2$, there always exists a partition $\mathscr{A}$ finer than these two, e.g.,

$$\mathscr{A} = \{A_1 \cap A_2 : A_i \in \mathscr{A}_i\}.$$

For $0 < \alpha \leqq 1$, the space of *α-Hölder* continuous functions is defined as

$$C^\alpha(X) = \{f: X \to \mathbb{C}: f \text{ is continuous and } \|f\|_{C^\alpha} < \infty\},$$

where

$$\|f\|_{C^\alpha} = \max\left\{\sup_{x \in X} |f(x)|, \sup_{\substack{x, y \in X \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}\right\}.$$

Let μ be any finite Borel measure on X. We recall that every finite Borel measure μ on a compact metric space X is *regular* in the sense that for any Borel set $E \subseteq X$ we have

$$\mu(E) = \inf\{\mu(G): G \text{ is open and } E \subseteq G\}$$

see, e.g., [4].

We shall work with the *class of kernels* $L_1(X, \mu; C^\alpha(X))$ formed by all Borel measurable kernels K: $X \times X \to \mathbb{C}$ such that the norm

$$\|K\|_{L_1(C^\alpha)} = \int_X \|K(t, \cdot)\|_{C^\alpha}\, d\mu(t)$$

is finite.

Note that if $K \in L_1(X, \mu; C^\alpha(X))$, then there is a non-negative function $L \in L_1(X, \mu)$ such that

$$|K(t, x) - K(t, y)| \leqq L(t)\, d(x, y)^\alpha \qquad \text{for all} \quad t, x, y \in X.$$

For $K \in L_1(X, \mu; C^\alpha(X))$, the *integral operator* with kernel K and measure μ is defined as

$$T_{K,\mu} f(x) = \int_X K(x, y)\, f(y)\, d\mu(y)$$

for $f \in L_1(X, \mu)$ and $x \in X$.

Subsequently, we are only interested in positive definite kernels, thus we always assume that K is Hermitian, i.e.,

$$K(x, y) = \overline{K(y, x)} \qquad \text{for all} \quad x, y \in X.$$

It is easily checked that the integral operator $T_{K,\mu}$ associated to such a kernel $K \in L_1(X, \mu; C^\alpha(X))$ defines a bounded operator in $L_1(X, \mu)$ and also in $L_\infty(X, \mu)$. Consequently, by the well-known Riesz-Thorin theorem, the operator

$$T_{K,\mu} \colon L_2(X, \mu) \to L_2(X, \mu)$$

is bounded.

Assume now that H is a (complex) Hilbert space and $S \in \mathscr{L}(H, H)$ is a compact operator. We denote by $(\lambda_n(S))$ the sequence of all *eigenvalues* of S counted according to their algebraic multiplicities and ordered with respect to decreasing absolute values,

$$|\lambda_1(S)| \geqq |\lambda_2(S)| \geqq \cdots \geqq 0.$$

If S has less than n eigenvalues, then we set

$$\lambda_n(S) = \lambda_{n-1}(S) = \cdots = 0.$$

The *singular numbers* of S are defined as

$$s_n(S) = \lambda_n([S^*S]^{1/2}).$$

Clearly,

$$s_n(S) = 0 \qquad \text{if} \quad \operatorname{rank}(S) < n.$$

Moreover, it holds

$$s_{n+m-1}(S_1 + S_2) \leqq s_n(S_1) + s_m(S_2)$$

see, e.g., [8].

An operator $S \in \mathscr{L}(H, H)$ is called *positive* if

$$(Sh, h) \geqq 0 \qquad \text{for all} \quad h \in H.$$

A Hermitian kernel K is called *positive definite* if $T_{K,\mu}$ is a positive operator in $L_2(X, \mu)$.

As usual, given two sequences (a_n), (b_n) of positive real numbers, we write $a_n = O_n(b_n)$ if $a_n \leqq cb_n$ for some $c > 0$ and all $n \in \mathbb{N}$, while $a_n \simeq b_n$ means that both $a_n = O(b_n)$ and $b_n = O(a_n)$. Moreover, we write $a_n = o(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$.

2. Eigenvalue Results

Before we can prove our main theorem we still need an auxiliary result which follows easily from elementary measure theory.

Lemma 1. *Let X be a compact metric space and let $d_n > 2\varepsilon_n(X)$ with $\lim_{n\to\infty} d_n = 0$. Then there are partitions $\mathscr{A}_n$ and $\mathscr{B}_n$ of X such that for all $n \in \mathbb{N}$ the following properties are satisfied:*

(1) $\operatorname{card}(\mathscr{A}_n) \leqq n$ *and* $\operatorname{diam}(\mathscr{A}_n) \leqq d_n$;

(2) $\mathscr{B}_{n+1} \prec \mathscr{B}_n \prec \mathscr{A}_n$.

Moreover, if we define

$$G_n := \operatorname{span}\{\chi_B : B \in \mathscr{B}_n\},$$

then

(3) *For any finite Borel measure μ on X, $\bigcup_{n=1}^{\infty} G_n$ is dense in the Hilbert space $L_2(X, \mu)$.*

Proof. We start by constructing the partitions $\mathscr{A}_n$. Since $r_n := d_n/2 > \varepsilon_n(X)$, it follows from the definition of the entropy numbers that there exist n balls $B_1, ..., B_n$ of radius r_n that cover X. Setting

$$A_1 := B_1 \qquad \text{and} \qquad A_i := B_i \setminus \bigcup_{j<i} A_j \qquad \text{for} \quad i = 2, ..., n,$$

we get a partition $\mathscr{A}_n = \{A_1, ..., A_n\}$ of X with property (1).

Next we find by induction the partitions $\mathscr{B}_n$. For $n = 1$ we take $\mathscr{B}_1 := \mathscr{A}_1$. If the partitions $\mathscr{B}_1, ..., \mathscr{B}_n$ have been already constructed, then we take as $\mathscr{B}_{n+1}$ any finite partition which is both finer than $\mathscr{A}_{n+1}$ and $\mathscr{B}_n$ (as we mentioned under Preliminaries, such partition always exists). Obviously (2) also holds.

It remains to prove (3). Let μ be any finite Borel measure on X. The step functions that take only finitely many values are dense in $L_2(X, \mu)$, see, e.g., [4]. Hence it suffices to show that for every Borel set $E \subseteq X$ and each $\varepsilon > 0$ there are $n \in \mathbb{N}$ and $g \in G_n$ with $\|\chi_E - g\|_{L_2} \leqq \varepsilon$.

Since μ is a regular measure, we can find an open set G such that $E \subseteq G$ and $\mu(G \backslash E) \leqq \varepsilon^2/4$. Next we define sets $F_n := \bigcup_{B \in \mathscr{B}_n, B \subseteq G} B$ and show that $G = \bigcup_{n=1}^{\infty} F_n$.

Given any $x \in G$ there is $\delta > 0$ with $B(x, \delta) \subseteq G$, since G is open. Choose $m \in \mathbb{N}$ with $d_m < \delta$. The point x belongs to some set B of the partition $\mathscr{B}_m$. Moreover, since $\mathscr{B}_m$ is finer than $\mathscr{A}_m$, we have $\operatorname{diam}(B) \leqq d_m < \delta$. This implies $x \in B \subseteq B(x, \delta) \subseteq G$, and therefore $x \in F_m$. Consequently $G \subseteq \bigcup_{n=1}^{\infty} F_n$. The other inclusion is obvious.

The sets F_n are increasing because the partitions $\mathscr{B}_n$ become finer with increasing n. This, together with the σ-additivity of μ, gives

$$\mu(G) = \lim_{n \to \infty} \mu(F_n).$$

Hence we can select an integer $n \in \mathbb{N}$ with

$$\mu(G \backslash F_n) \leqq \varepsilon^2/4.$$

By definitions of F_n and G_n, the function $g = \chi_{F_n}$ belongs to G_n. This finally yields

$$\begin{aligned} \|\chi_E - g\|_{L_2} &\leqq \|\chi_E - \chi_G\|_{L_2} + \|\chi_G - g\|_{L_2} \\ &= \mu(G \backslash E)^{1/2} + \mu(G \backslash F_n)^{1/2} \leqq \varepsilon. \end{aligned}$$

Thus (3) is satisfied as well, and the proof is finished. ∎

After this preparation we can pass to our main result on the asymptotic eigenvalue behaviour of integral operators whose kernels are positive definite and satisfy an integrated α-Hölder condition.

THEOREM 2. *Let X be a compact metric space equipped with a finite Borel measure μ, let $0 < \alpha \leqq 1$, and assume that $\varepsilon_{2n}(X) \asymp \varepsilon_n(X)$. Then, for every positive definite kernel $K \in L_1(X, \mu; C^\alpha(X))$, one has*

$$\lambda_n(T_{K,\mu}) = O(n^{-1} \varepsilon_n(X)^\alpha).$$

Proof. We shall write simply T instead of $T_{K,\mu}$ and L_2 instead of $L_2(X, \mu)$. We shall also use the notation introduced in Lemma 1.

Let P_n be the orthogonal projection in L_2 onto the finite-dimensional subspace G_n constructed in Lemma 1. Since the G_n's are increasing and their union is dense in L_2, we can find an orthonormal basis $(e_j)_{j \in \mathbb{N}}$ such that

$$G_n = \operatorname{span}\{e_1, ..., e_{j_n}\}.$$

For any operator S in L_2, we have

$$\operatorname{trace}(SP_n) = \sum_{j=1}^{j_n} (Se_j, e_j).$$

Moreover

$$\operatorname{trace}(S) = \sum_{j=1}^{\infty} (Se_j, e_j)$$

provided the series converges. For positive S, all summands are non-negative, so the convergence of the series is equivalent to

$$\sup_n \sum_{j=1}^{j_n} (Se_j, e_j) = \sup_n \operatorname{trace}(SP_n) < \infty.$$

In this case,

$$\operatorname{trace}(S) = \sup_n \operatorname{trace}(SP_n).$$

Now let Q be the orthogonal projection onto

$$E := \operatorname{span}\{\chi_A : A \in \mathscr{A}_m\},$$

where m is some fixed integer and $\mathscr{A}_m$ is the partition from Lemma 1. We want to estimate from above

$$\operatorname{trace}((I-Q)\, T(I-Q)).$$

Obviously $(I-Q)\, T(I-Q)$ is a positive operator in L_2, hence the preceding observaion applies. So, let us fix $n \geqq m$, and let $A_1, ..., A_k$ and $B_1, ..., B_r$ be those sets having positive μ-measure in $\mathscr{A}_m$ and $\mathscr{B}_n$, respectively. Then the functions

$$f_i := \mu(A_i)^{-1/2} \chi_{A_i}, i = 1, ..., k, \qquad \text{and} \qquad g_j := \mu(B_j)^{-1/2} \chi_{B_j}, j = 1, ..., r,$$

form orthonormal bases in E and G_n, respectively. Moreover, we have $E \subseteq G_n$ because $\mathscr{B}_n$ is finer than $\mathscr{A}_m$. Therefore

$$QP_n = P_n Q = Q.$$

Observing that

$$\sum_{i=1}^{k} f_i(t) \int_X f_i(y)\, d\mu(y) = 1 \qquad \text{a.e.}$$

and

$$\sum_{j=1}^{r} g_j(t) \int_X g_j(x)\, d\mu(x) = 1 \qquad \text{a.e.}$$

we obtain

$$\begin{aligned}
\operatorname{trace}((I-Q)\,T(I-Q)P_n) &= \operatorname{trace}(T(I-Q)\,P_n(I-Q)) \\
&= \operatorname{trace}(T(P_n - Q)) \\
&= \operatorname{trace}(TP_n) - \operatorname{trace}(TQ) \\
&= \sum_{j=1}^{r} \int_X \int_X K(t,x)\, g_j(x)\, g_j(t)\, d\mu(x)\, d\mu(t) \\
&\quad - \sum_{i=1}^{k} \int_X \int_X K(t,y)\, f_i(y)\, f_i(t)\, d\mu(y)\, d\mu(t) \\
&= \sum_{i=1}^{k} \sum_{j=1}^{r} \int_X \int_X \int_X [K(t,x) - K(t,y)]\, g_j(x)\, g_j(t) \\
&\quad \times f_i(y)\, f_i(t)\, d\mu(x)\, d\mu(y)\, d\mu(t).
\end{aligned}$$

Since every B_j is contained in some A_i, we can split the index set $\{1, ..., r\}$ into disjoint subsets

$$I_i = \{j : B_j \subseteq A_i\}, \qquad i = 1, ..., k.$$

For fixed i, the integrand is only non-zero when

$$y, t \in A_i \qquad \text{and} \qquad x, t \in B_j \qquad \text{for some } j \in I_i.$$

But in this case, it holds $x \in A_i$ as well. Whence the assumption on the kernel implies

$$\begin{aligned}
|K(t,x) - K(t,y)| &\leqq L(t)\, d(x,y)^\alpha \\
&\leqq L(t)[\operatorname{diam}(A_i)]^\alpha
\end{aligned}$$

for some non-negative function $L \in L_1(X, \mu)$. By Lemma 1

$$\operatorname{diam}(A_i) \leqq d_m,$$

where $d_m > 2\varepsilon_m(X)$ can be chosen arbitrarily. In addition, the properties of the partitions of Lemma 1 give

$$\mu(A_i) = \sum_{j \in I_i} \mu(B_j), \qquad i = 1, ..., k,$$

and

$$\bigcup_{i=1}^{k} \bigcup_{j \in I_i} B_j = X.$$

All this together yields the estimate

$$\begin{aligned} \operatorname{trace}((I-Q)\, T(I-Q)P_n) &\leq d_m^{\alpha} \sum_{i=1}^{k} \sum_{j \in I_i} \int_{B_j} \int_{A_i} \int_{B_j} L(t)\, d\mu(x) \\ &\quad \times d\mu(y)\, d\mu(t)\, \mu(A_i)^{-1}\, \mu(B_j)^{-1} \\ &= d_m^{\alpha} \sum_{i=1}^{k} \sum_{j \in I_i} \int_{B_j} L(t)\, d\mu(t) \\ &= d_m^{\alpha} \|L\|_{L_1}. \end{aligned}$$

On the other hand, taking into account that

$$\operatorname{rank}[T - (I-Q)\, T(I-Q)] \leq 2k \leq 2m,$$

we have

$$\begin{aligned} s_{3m}(T) &\leq s_m((I-Q)\, T(I-Q)) + s_{2m+1}(T - (I-Q)\, T(I-Q)) \\ &= s_m((I-Q)\, T(I-Q)). \end{aligned}$$

Consequently,

$$\begin{aligned} m s_{3m}(T) &\leq m s_m((I-Q)\, T(I-Q)) \\ &\leq \sum_{j=1}^{\infty} s_j((I-Q)\, T(I-Q)) \\ &= \operatorname{trace}((I-Q)\, T(I-Q)) \\ &= \sup_n \operatorname{trace}((I-Q)\, T(I-Q)P_n) \leq d_m^{\alpha} \|L\|_{L_1}. \end{aligned}$$

Letting $d_m \to 2\varepsilon_m(X)$ and observing that $\lambda_{3m}(T) = s_{3m}(T)$, since T is a positive operator in L_2, we obtain

$$\lambda_{3m}(T) \leq m^{-1} 2^{\alpha} \varepsilon_m(X)^{\alpha} \|L\|_{L_1}.$$

Finally, the assumption $\varepsilon_{2n}(X) \simeq \varepsilon_n(X)$ yields the desired asymptotic behaviour

$$\lambda_n(T) = O(n^{-1}\varepsilon_n(X)^\alpha). \quad \blacksquare$$

Remark 3. The assumption $\varepsilon_{2n}(X) \simeq \varepsilon_n(X)$ excludes, roughly speaking, too fast decay of the entropy numbers of X. This condition is not very restrictive; e.g., it is satisfied for every connected compact metric space (see [5, Lemma 3]), or if $\varepsilon_n(X) \simeq n^{-\beta}(\log n)^\gamma$ for some $\beta > 0$ and $\gamma \in \mathbb{R}$. In the last section of the paper we give examples of such compact metric spaces which are totally disconnected.

In the proof of Theorem 1 we have note used the compactness of X but only the behaviour of its entropy numbers. Therefore, since the entropy numbers of any bounded Borel set $\Omega \subseteq \mathbb{R}^N$ with non-empty interior are of the same asymptotic order as those of the unit cube $[0, 1]^N$, i.e., $\varepsilon_n(\Omega) \simeq n^{-1/N}$, we derive as an immediate consequence

THEOREM 4. *Let μ be a Borel measure on a bounded Borel set $\Omega \subseteq \mathbb{R}^N$ with non-empty interior and let $K \in L_1(\Omega, \mu; C^\alpha(\Omega))$ be a positive definite kernel. Then*

$$\lambda_n(T_{K,\mu}) = O(n^{-\alpha/N-1}).$$

This theorem extends Reade's result mentioned in the Introduction to multidimensional domains and arbitrary Borel measures. Note that global continuity of the kernel is not required, although this extra assumption was necessary for Reade's original proof [9] (and also for the Cochran and Lukas one [3]).

3. OPTIMALITY OF THE EIGENVALUE ESTIMATES

In this section we comment on the sharpness of the eigenvalue results established in Theorems 2 and 4.

The following theorem shows that the asymptotic order of the eigenvalues established in Theorem 2 is the best possible.

THEOREM 5. *Let X be a compact metric space satisfying $\varepsilon_{2n}(X) \simeq \varepsilon_n(X)$, and let $0 < \alpha \leqq 1$. Then for every sequence (a_n) of positive real numbers with $a_n = o(n^{-1}\varepsilon_n(X)^\alpha)$ there are a finite Borel measure μ on X and a positive definite kernel $K \in L_1(X, \mu; C^\alpha(X))$ such that*

$$\overline{\lim_{n \to \infty}} \lambda_n(T_{K,\mu})/a_n = \infty.$$

Proof. First we show that (a_n) can be majorized by a sequence (A_n) such that still $A_n = o(n^{-1}\varepsilon_n(X)^\alpha)$ and moreover $A_{2n} \simeq A_n$. This latter property is essential for our proof.

By assumption

$$a_n \leqq c_n n^{-1} \varepsilon_n(X)^\alpha \qquad \text{with} \quad \lim_{n \to \infty} c_n = 0.$$

Defining

$$C_1 := \sup_{k \geqq 1} c_k, \qquad C_{2n} = C_{2n+1} := \max\left(\frac{C_n}{2}, \sup_{k \geqq 2n} c_k\right) \qquad \text{for} \quad n \geqq 1$$

we obtain a sequence which clearly satisfies

$$c_n \leqq C_n \qquad \text{and} \qquad C_{2n} \geqq C_n/2,\ n = 1, 2, \ldots.$$

By induction one easily verifies that (C_n) is decreasing. Hence $\lim_{n \to \infty} C_n$ exists, and the estimate

$$\lim_{n \to \infty} C_n = \lim_{n \to \infty} C_{2n} \leqq \tfrac{1}{2} \lim_{n \to \infty} C_n + \lim_{n \to \infty} \sup_{k \geqq 2n} c_k = \tfrac{1}{2} \lim_{n \to \infty} C_n$$

shows that this limit is zero. Then taking

$$A_n := C_n n^{-1} \varepsilon_n(X)^\alpha$$

we get the desired sequence. Obviously $A_n \geqq a_n$ and $A_n = o(n^{-1}\varepsilon_n(X)^\alpha)$. Moreover $A_{2n} \simeq A_n$ because the sequences (C_n), (n^{-1}) and $(\varepsilon_n(X))$ enjoy this property.

So without loss of generality we may assume that $a_{2n} \simeq a_n$. Put $b_n := (na_n)^{1/\alpha}$. Then one has

$$\lim_{n \to \infty} \varepsilon_n(X)/b_n = \infty \qquad \text{and} \qquad b_{2n} \simeq b_n.$$

The following construction is based on ideas taken from [5, Theorem 3]. For $k = 1, 2, \ldots$, we find inductively positive integers n_k, real numbers ε_k, $0 < \varepsilon_k < 1$, and subsets X_k, Y_k of X with the following properties:

(1) $n_k > n_{k-1}$ and $\varepsilon_{k-1} \geqq \varepsilon_k \geqq k^{5/\alpha} b_{n_k}$;

(2) X_k and Y_k are closed subsets of Y_{k-1};

(3) X_k is a $2\varepsilon_k$-distant subset consisting of n_k elements;

(4) $d(X_k, Y_k) \geqq \varepsilon_k$;

(5) $\overline{\lim}_{n \to \infty} \varepsilon_n(Y_k)/b_n = \infty$.

To prove this, assume that for $1 \leqq j < k$ we have already found n_j, ε_j, X_j, and Y_j with properties (1) to (5). (In the first step of induction we can argue in the same way but starting with X instead of Y_{k-1}.) Using (5) and the fact that $b_{2n} \simeq b_n$, we can find an integer $n_k > n_{k-1}$ such that

$$k^{5/\alpha} b_{n_k} \leqq \varepsilon_k := \varepsilon_{2n_k}(Y_{k-1})/3.$$

Moreover, compactness of Y_{k-1} implies that its entropy numbers tend to zero. Hence we may also assume that $\varepsilon_k \leqq \varepsilon_{k-1}$. Thus (1) is satisfied. Let now M be a maximal $2\varepsilon_k$-distant subset of Y_{k-1}. Then

$$Y_{k-1} \subseteq \bigcup_{x \in M} B(x, 2\varepsilon_k),$$

and since $2\varepsilon_k < \varepsilon_{2n_k}(Y_{k-1})$, it also follows that $\operatorname{card}(M) \geqq 2n_k$. So we can select two disjoint subsets M_1 and M_2 of M, each one containing n_k elements. The sets

$$Z_i := \bigcup_{x \in M_i} B(x, \varepsilon_k), \qquad i = 1, 2,$$

are disjoint as well. Whence

$$Y_{k-1} = (Y_{k-1} \backslash Z_1) \cup (Y_{k-1} \backslash Z_2).$$

Using again (5), the trivial observation

$$\varepsilon_{2n}(Y_{k-1}) \leqq \max_{i=1,2} \varepsilon_n(Y_{k-1} \backslash Z_i), \qquad n \in \mathbb{N},$$

and the fact that $b_{2n} \simeq b_n$, we obtain that for at least one i, say $i = 1$,

$$\overline{\lim_{n \to \infty}}\; \varepsilon_n(Y_{k-1} \backslash Z_1)/b_n = \infty$$

still holds. Consequently, if we set

$$X_k := M_1 \qquad \text{and} \qquad Y_k := Y_{k-1} \backslash Z_1,$$

the conditions (2) to (5) are satisfied.

Next we construct the measure μ and the kernel K.

Put

$$X_k = \{x_{k,j} \colon j = 1, ..., n_k\}, \qquad k \in \mathbb{N},$$

and define μ as the point measure assigning to $x_{k,j}$, $j = 1, ..., n_k$, $k \in \mathbb{N}$, the mass $k^{-2} n_k^{-1}$. Clearly μ is a finite Borel measure on X.

In order to construct the kernel, consider the functions (see [5])

$$f_{k,j}(x) := [\max(0, 1 - d(x, x_{k,j})/\varepsilon_k)]^\alpha, \qquad x \in X.$$

Elementary computations show that for any $k \in \mathbb{N}$ and any sequence of scalars $(\xi_j)_{j=1}^{n_k}$ it holds

$$\left\| \sum_{j=1}^{n_k} \xi_j f_{k,j} \right\|_{C^\alpha} \leqq 2\varepsilon_k^{-\alpha} \max_{1 \leqq j \leqq n_k} |\xi_j|. \tag{*}$$

Define now

$$K(x, y) := \sum_{k=1}^{\infty} k^{-2} \varepsilon_k^\alpha \sum_{j=1}^{n_k} f_{k,j}(x) f_{k,j}(y).$$

For each $x \in X$ and $k \in \mathbb{N}$, it follows from (*) that

$$\left\| \sum_{j=1}^{n_k} f_{k,j}(x) f_{k,j} \right\|_{C^\alpha} \leqq 2\varepsilon_k^{-\alpha} \max_{1 \leqq j \leqq n_k} |f_{k,j}(x)| \leqq 2\varepsilon_k^{-\alpha}.$$

Thus, for every $x \in X$, we get

$$\|K(x, \cdot)\|_{C^\alpha} \leqq 2 \sum_{k=1}^{\infty} k^{-2} = c < \infty,$$

and consequently

$$\|K\|_{L_1(C^\alpha)} = \int_X \|K(x, \cdot)\|_{C^\alpha} \, d\mu(x) \leqq c\mu(X) < \infty.$$

Moreover, K is positive definite because, given any $f \in L_2(X, \mu)$, we have

$$(T_{K,\mu} f, f) = \sum_{k=1}^{\infty} k^{-2} \varepsilon_k^\alpha \sum_{j=1}^{n_k} |(f, f_{k,j})|^2 \geqq 0.$$

To complete the proof, we need to show that

$$\overline{\lim_{n \to \infty}} \lambda_n(T_{K,\mu})/a_n = \infty.$$

According to the definition of $f_{k,j}$, we see that

$$f_{k,j}(x_{r,s}) = \begin{cases} 1 & \text{if } k = r \text{ and } j = s, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for $k \in \mathbb{N}$ and $j = 1, \ldots, n_k$, we have

$$T_{K,\mu} f_{k,j} = k^{-4} n_k^{-1} \varepsilon_k^\alpha f_{k,j}.$$

Whence, using (1) and recalling that $b_n = (na_n)^{1/\alpha}$, we arrive at the inequality

$$\lambda_{n_k}(T_{K,\mu}) \geqq k^{-4} n_k^{-1} \varepsilon_k^{\alpha} \geqq k n_k^{-1} b_{n_k}^{\alpha} = k a_{n_k}.$$

Since this estimate holds for the strictly increasing sequence (n_k), we finally obtain the desired assertion

$$\varlimsup_{n \to \infty} \lambda_n(T_{K,\mu})/a_n = \infty. \quad \blacksquare$$

Theorem 5 also implies that the asymptotic behaviour of the eigenvalues established in Theorem 4 is the best possible. In fact, for the case of bounded domains in $\mathbb{R}^N$, the optimality even holds in a stronger sense, namely without constructing a suitable measure but using the most natural measure on $\mathbb{R}^N$, i.e., the Lebesgue measure.

Indeed, consider the N-dimensional unit cube $[0, 1]^N$ equipped with the Lebesgue measure μ. It was proved in [7, Theorem 5] that, given any $0 < \alpha \leqq 1$, there exists a positive definite continuous kernel K such that

$$\sup_{\substack{t, x, y \in X \\ x \neq y}} \frac{|K(t, x) - K(t, y)|}{d(x, y)^{\alpha}} < \infty \qquad [\text{therefore } K \in L_1([0, 1]^N; C^{\alpha}([0, 1]^N))]$$

and

$$\lambda_n(T_{K,\mu}) \asymp n^{-\alpha/N - 1}.$$

Whence the eigenvalue estimate cannot be improved.

4. Examples of Compact Metric Spaces with Regular Entropy Behaviour

We have already mentioned in Remark 3 that the condition $\varepsilon_{2n}(X) \asymp \varepsilon_n(X)$ which we need for our results, is satisfied for every connected compact metric space, or if $\varepsilon_n(X) \asymp n^{-\beta}(\log n)^{\gamma}$. In this final section we give fairly easy examples of totally disconnected compact metric spaces X such that

$$\varepsilon_n(X) \asymp n^{-\beta}(\log n)^{\gamma}$$

for given $\beta > 0$ and $\gamma \in \mathbb{R}$, or $\beta = 0$ and $\gamma < 0$. Similar examples can be found in [6], but only for a very special choice of β and γ.

First we describe a construction which is quite common in geometric measure theory and in the theory of fractals. The basic model of this construction is the Cantor set.

Let $q=(q_n)$ be any sequence of positive real numbers such that for some $r>0$

$$r \leqq q_n \leqq 1/3, \qquad n=1, 2, \ldots.$$

Start with the unit interval $[0, 1]$ and remove in the first step an open interval, such that two closed intervals of length q_1 remain. Then proceed inductively, but at the nth step take the ratio q_n.

So, after n steps, one has 2^n closed intervals $I_{n,j}$, $1 \leqq j \leqq 2^n$, of length $q_1 \cdots q_n$ with mutual distance $\geqq q_1 \cdots q_n$ (since all $q_n \leqq 1/3$). Put

$$E_n := \bigcup_{j=1}^{2^n} I_{n,j} \qquad \text{and} \qquad C_{1,q} := \bigcap_{n=1}^{\infty} E_n.$$

Note that never endpoints of intervals are removed. Therefore $C_{1,q}$ contains 2^{n+1} points (namely the endpoints of the 2^n intervals $I_{n,j}$) which have mutual distance $\geqq q_1 \cdots q_n$. Moreover, the 2^{n+1} (closed) balls centered at these points with radius $q_1 \cdots q_n/2$ cover E_n and hence also $C_{1,q}$.

Carrying out this construction in $\mathbb{R}^N$ (with the sup-metric), i.e., setting

$$C_{N,q} := \bigcap_{n=1}^{\infty} \underbrace{E_n \times \cdots \times E_n}_{N\text{-times}},$$

we get $2^{(n+1)N}$ points with the same properties as above. Whence,

$$\frac{q_1 \cdots q_n}{2} \leqq \varepsilon_{2^{nN}}(C_{N,q}) \leqq \frac{q_1 \cdots q_{n-1}}{2} \leqq \frac{1}{r} \cdot \frac{q_1 \cdots q_n}{2}$$

that means

$$\varepsilon_{2^{nN}}(C_{N,q}) \asymp q_1 \cdots q_n. \tag{$\ast\ast$}$$

Let now (a_n) be any sequence of positive real numbers such that for $n>n_0$ it holds

$$a \leqq a_{kn}/a_n \leqq A \qquad \text{and} \qquad a_{n+1} \leqq a_n,$$

where $0<a<A<1$ are constants and $k \geqq 2$ is some integer. This implies that for appropriate $r>0$ and $N \in \mathbb{N}$

$$r \leqq \frac{a_{2^{nN}}}{a_{2^{(n-1)N}}} \leqq 1/3 \qquad \text{for all} \quad n>n_0.$$

Let $q = (q_n)$ be the sequence defined by

$$q_1 = \cdots = q_{n_0} = 1/3 \qquad \text{and} \qquad q_n = \frac{a_{2^{nN}}}{a_{2^{(n-1)N}}} \qquad \text{for} \quad n > n_0,$$

and let $C_{N,q}$ be the N-dimensional set of Cantor type associated to q. Then it follows from $(\ast)$ that

$$\varepsilon_{2^{nN}}(C_{N,q}) \simeq q_1 \cdots q_n = (1/3)^{n_0} \frac{a_{2^{nN}}}{a_{2^{n_0 N}}} \simeq a_{2^{nN}},$$

and consequently

$$\varepsilon_n(C_{N,q}) \simeq a_n.$$

Clearly, given any $\beta > 0$ and $\gamma \in \mathbb{R}$, the sequence $(n^{-\beta}(\log n)^\gamma)$ satisfies the properties required on (a_n). So we have in particular

PROPOSITION 6. *Let $\beta > 0$ and $-\infty < \gamma < \infty$. Then there exists a set of Cantor type $C_{N,q}$ such that*

$$\varepsilon_n(C_{N,q}) \simeq n^{-\beta}(\log n)^\gamma,$$

Note that this result still holds for the more general class of sequences

$$(n^{-\beta}(\log n)^\gamma (\log \log n)^\delta),$$

and one can even add more iterated logarithms.

Finally, we discuss the case $\beta = 0$. This time our example is similar to the Hilbert cube.

PROPOSITION 7. *Given any $\gamma > 0$*, let

$$X_\gamma = \{\xi = (\xi_k) \in l_\infty : |\xi_k| \leqq k^{-\gamma}\}.$$

Then

$$\varepsilon_n(X_\gamma) \simeq (\log n)^{-\gamma}.$$

Proof. In order to produce a $2n^{-\gamma}$-distant set in X_γ, choose for each k, $1 \leqq k \leqq n$, a $2n^{-\gamma}$-distant set $\{\xi_j^{(k)}\}_{j=1}^{m_k}$ in $[-k^{-\gamma}, k^{-\gamma}]$ such that $m_k \geqq (n/k)^\gamma$. Clearly the set

$$M = \{\xi = (\xi_{j_1}^{(1)}, ..., \xi_{j_n}^{(n)}, 0, 0, ...) : 1 \leqq j_k \leqq m_k\}$$

is $2n^{-\gamma}$-distant in X_γ. Moreover, for n sufficiently large, we have

$$\operatorname{card}(M) = \prod_{k=1}^{n} m_k \geqq (n^n/n!)^\gamma \geqq 2^{n\gamma}.$$

Hence, if $2^{(n-1)\gamma} \leqq m < 2^{n\gamma}$, it is not possible to cover X_γ by m balls of radius $n^{-\gamma}$, and therefore

$$\varepsilon_m(X_\gamma) \geqq n^{-\gamma} \simeq (\log m)^{-\gamma}.$$

This gives the lower estimate. The upper one can be established with similar arguments. ∎

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